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Thermodynamics and Fluid Mechanics 2

Fluids Topic 1: Navier-Stokes equations

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Intro to Fluid Mechanics

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Timetable

Week	w/b	Lecture	Seminar
7	31 Oct	T1: Navier-Stokes equations	T1 lecture
8	7 Nov	T2: Boundary layer flows	T1
9	14 Nov	T2: Boundary layer flows	T2
10	21 Nov	T3: Lift & drag	T2
11	28 Nov	T3: Lift & drag	T3
12	05 Dec	Don Giddings - thermo	T3
20	30 Jan	T4: Dimensional analysis	T4
21	06 Feb	T4: Dimensional analysis	T4
27	20 Mar	T5: Turbomachinery	T5
28	27 Mar	T5: Turbomachinery	T5
33	01 May	T6: Compressible flows	T6
34	08 May	T6: Compressible flows	T6
35	15 May	Revision	

Resources:

- F. White, Fluid Mechanics

  https://nusearch.nottingham.ac.uk/permalink/f/1m5tnd/44NOTUK_ALMA2190232730005561

- B. S. Massey, Mechanics of Fluids

  https://nusearch.nottingham.ac.uk/permalink/f/1m5tnd/44NOTUK_ALMA2188819970005561

- Module notes (on Moodle)

You will find pdf versions of the books above quite easily just googling them.

Topic 1 – Navier-Stokes equations

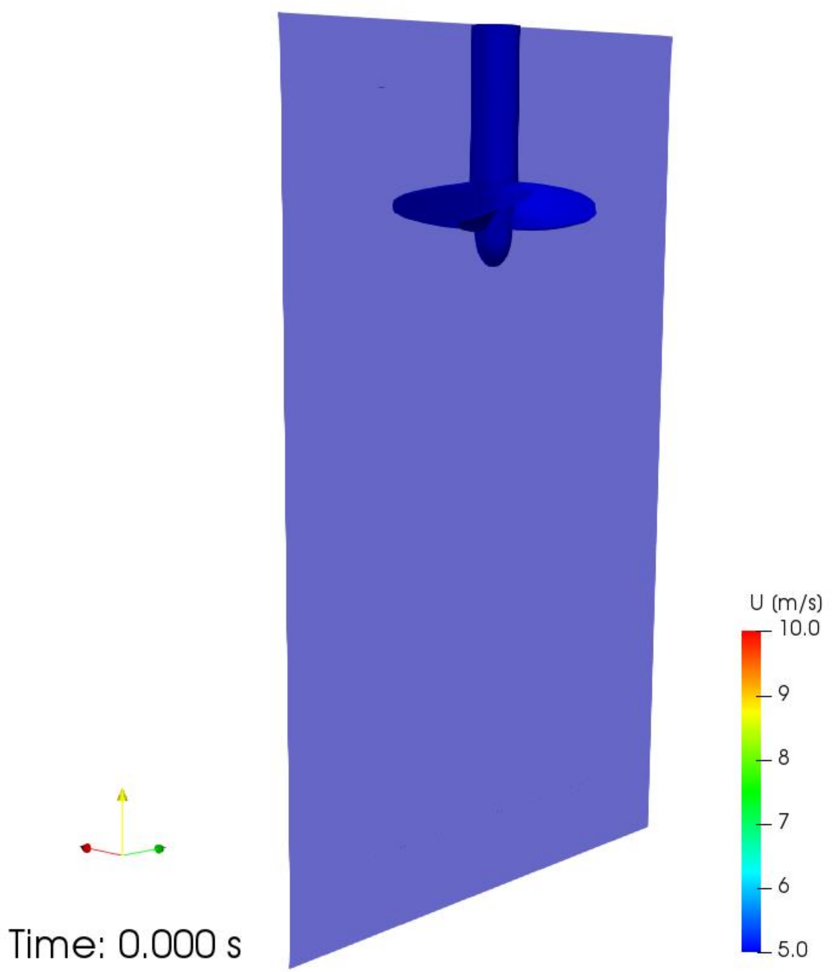
Topic 1 can be studied in F. White, Ch. 4

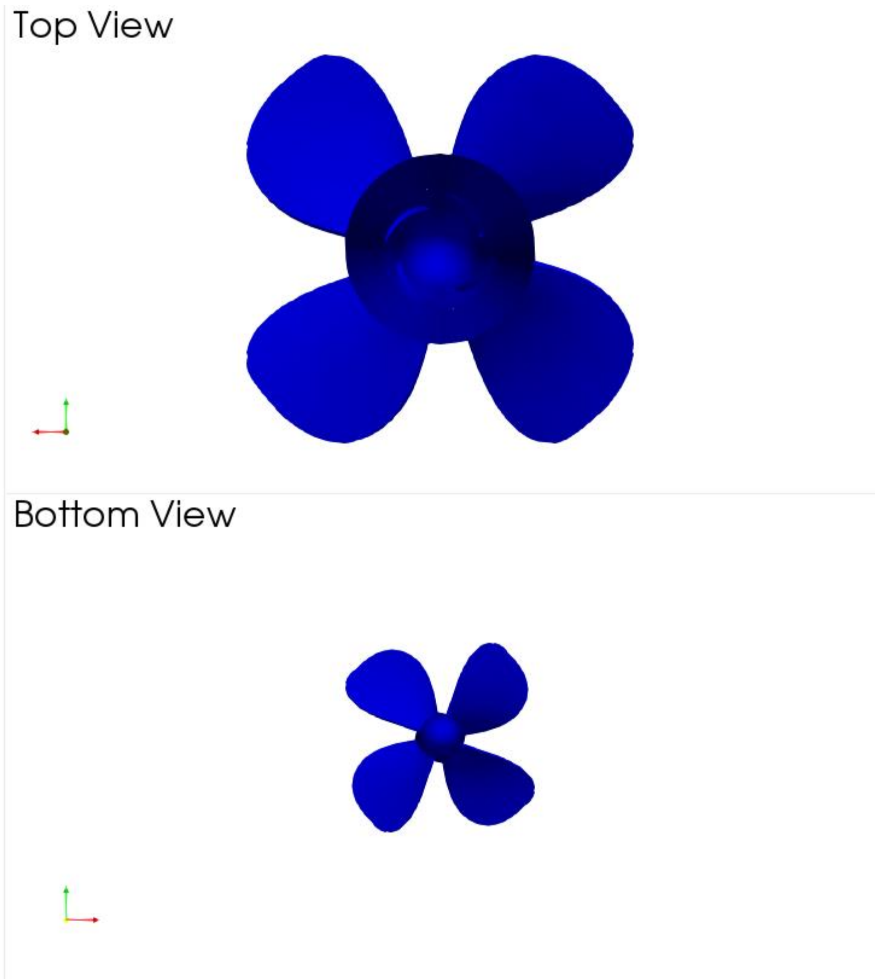
- The velocity field
- Conservation of mass
- Conservation of momentum
- Forces acting on a fluid in motion
- Representation of the surface stresses
- Newtonian fluids
- Navier-Stokes equations
- Boundary conditions
- Analytical solutions for N-S equations (seminar)

Learning outcomes: apply the concept of control volume to describe fluid flow; know how to derive mass/momentum eqs; know the constitutive laws of Newtonian fluids; know how to apply the N-S eqs to simple cases and obtain analytical solutions.

Motivation

Fluid mechanics is governed by the N-S eqs; if we can solve them, for example using CFD, we can obtain the fluid flow, e.g. **flow generated by a propeller**

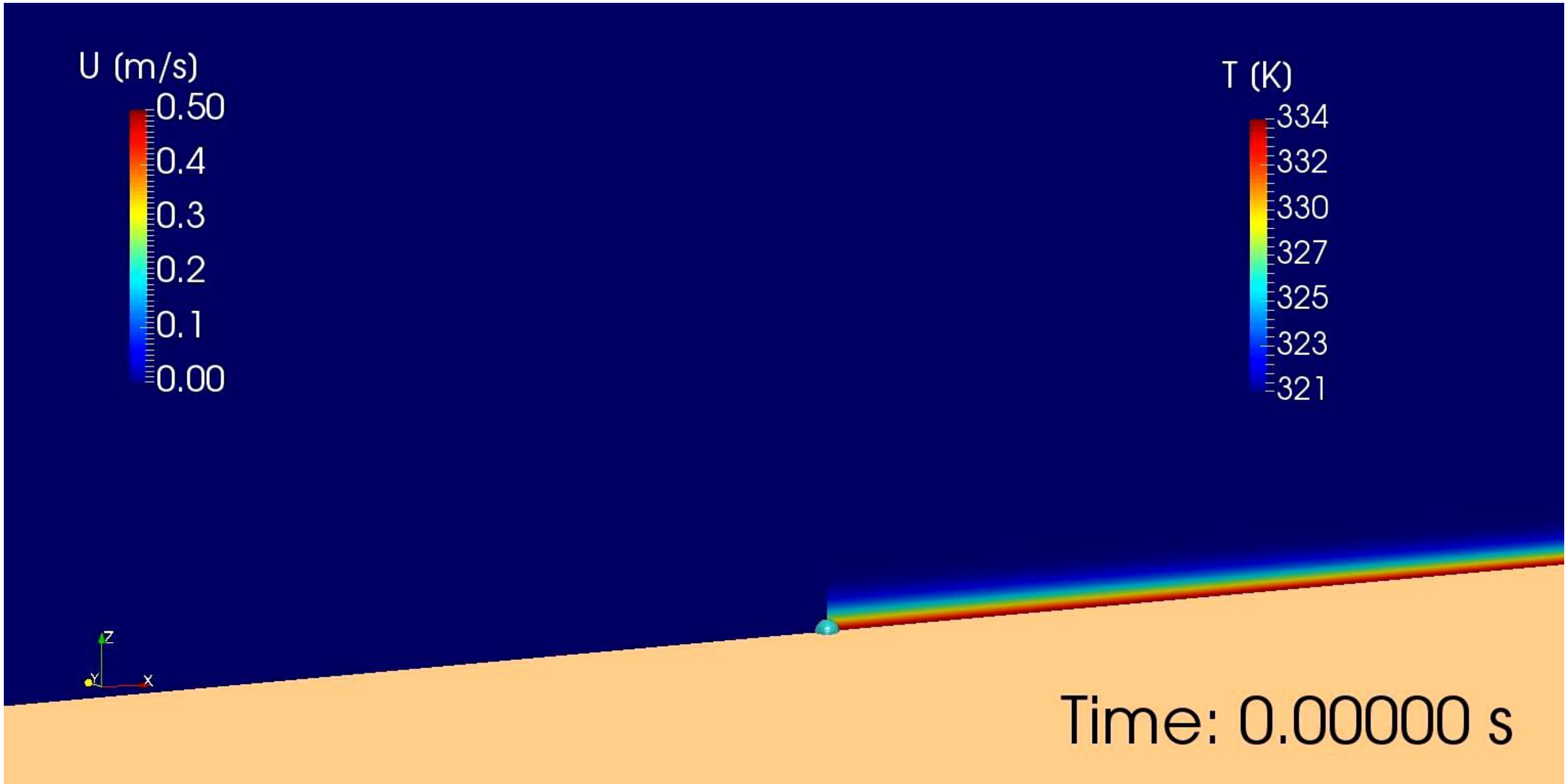




CFD simulation done with the open-source package OpenFOAM: https://openfoam.com/

Motivation

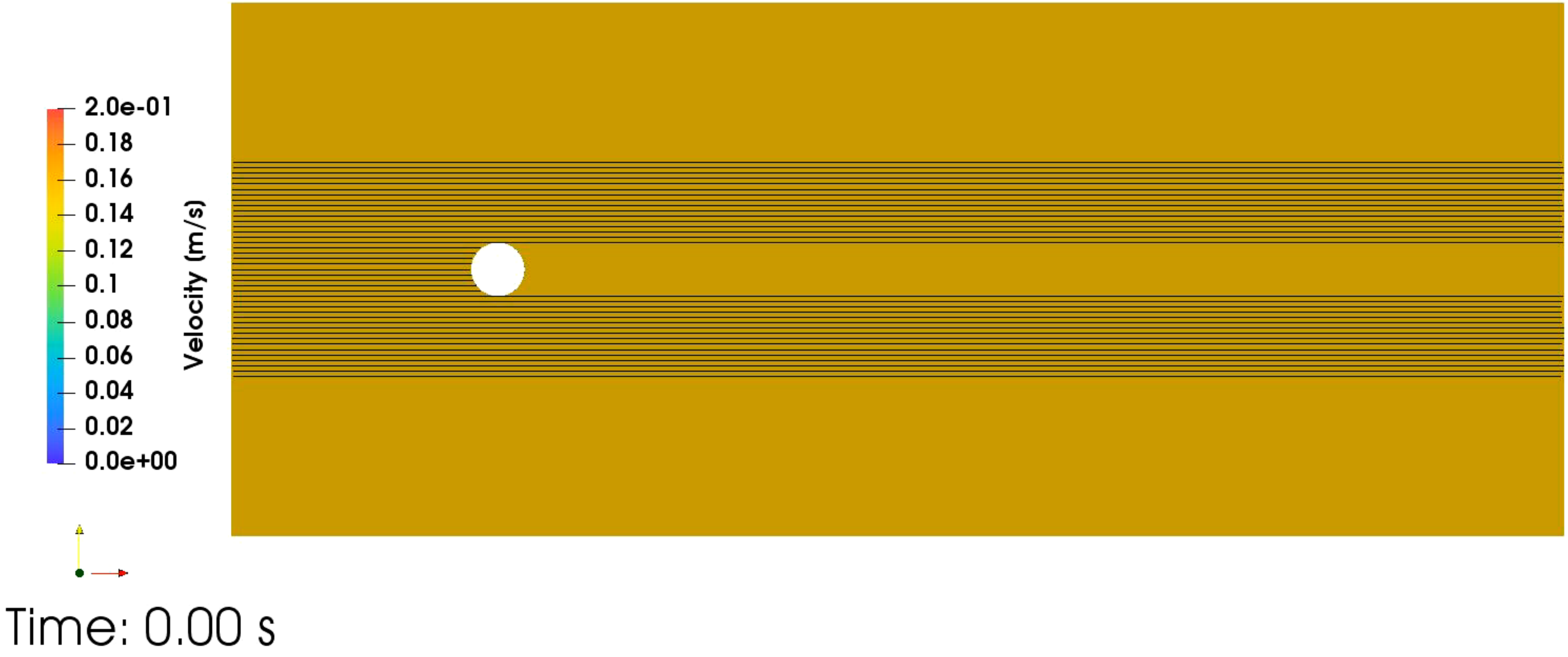
Fluid mechanics is governed by the N-S eqs; if we can solve them, for example using CFD, we can obtain the fluid flow, e.g. **boiling of a refrigerant on a hot surface**



CFD simulation done with the open-source package OpenFOAM: https://openfoam.com/

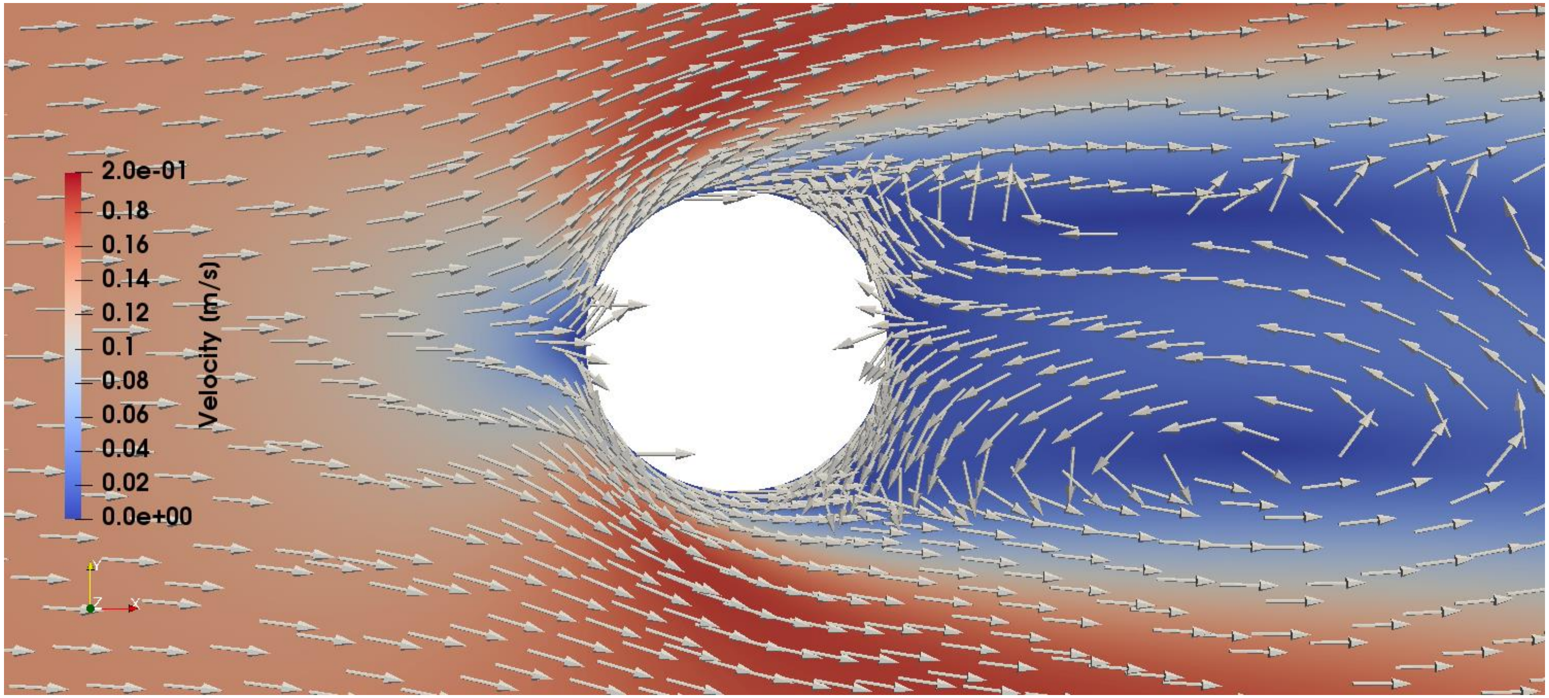
Motivation

Fluid mechanics is governed by the N-S eqs; if we can solve them, for example using CFD, we can obtain the fluid flow, e.g. **laminar flow past a cylinder**



CFD simulation done with the open-source package OpenFOAM: https://openfoam.com/

Velocity of a fluid



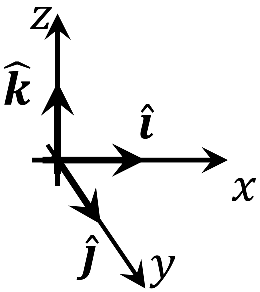
Each velocity vector provides an information on the local velocity field:

$$\boldsymbol{V}(x,y,z,t) = u(x,y,z,t)\hat{\boldsymbol{\imath}} + v(x,y,z,t)\hat{\boldsymbol{\jmath}} + w(x,y,z,t)\widehat{\boldsymbol{k}}$$

$u(x,y,z,t)$: velocity component along x

$v(x,y,z,t)$: velocity component along y

$w(x,y,z,t)$: velocity component along z

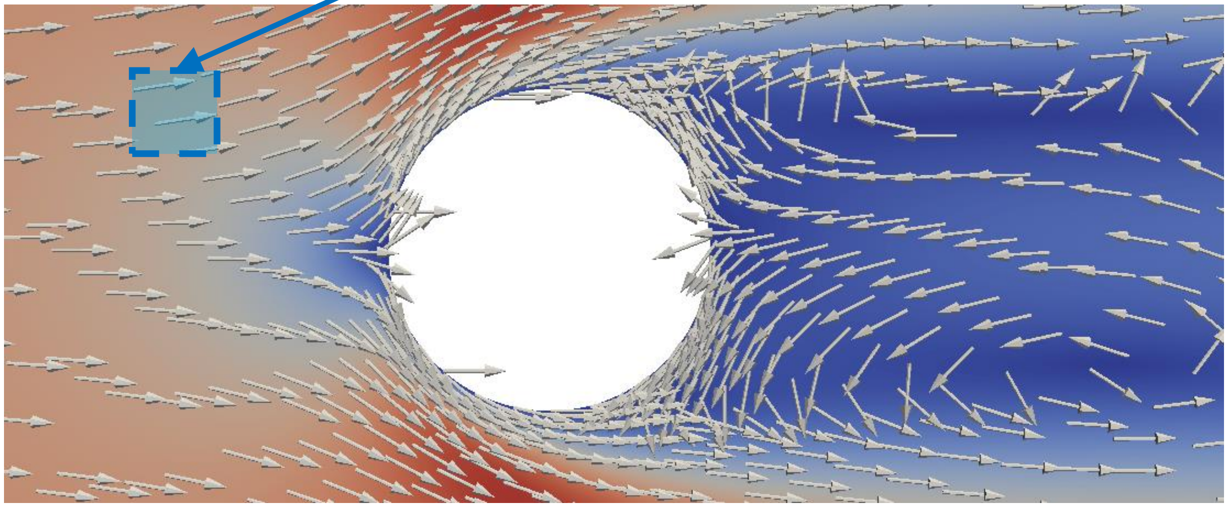


Navier-Stokes equations

But which equations do govern the flow field?

- Conservation of mass (aka continuity equation)
- Conservation of momentum
- Conservation of energy (not treated here)

Conservation of mass: we consider a generic control volume within the flow region, where the fluid can pass through ⇒ the temporal variation of the mass of fluid contained in the control volume must be equal to the net inflow/outflow through its surface.

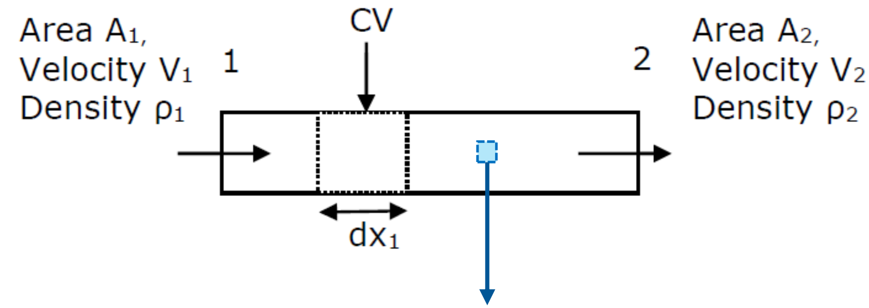


Conservation of mass

Let's restart from TF1: lecture notes page 77-78, flow in a pipe

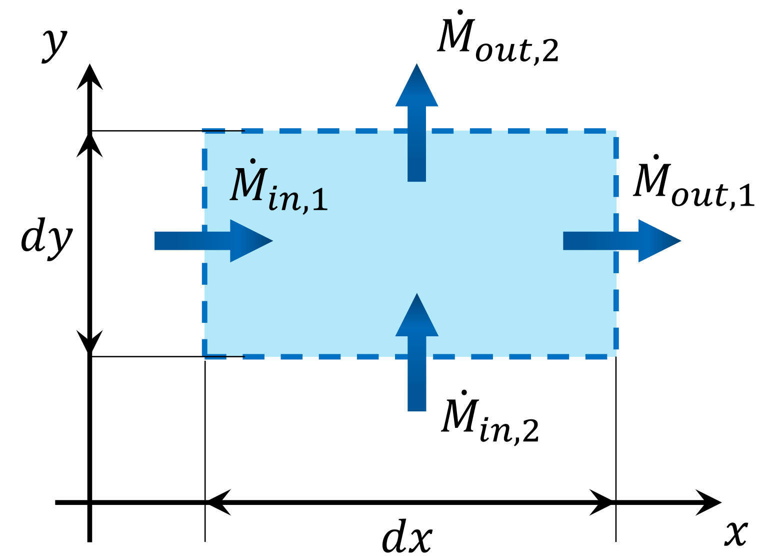
[…] if there are several inlet and outlet streams the equations generalize to:

$$\frac{dM}{dt} = \sum \dot{M}_{in} - \sum \dot{M}_{out} \qquad units: \frac{kg}{s}$$



or:

$$\frac{dM}{dt} = \sum (\rho A V)_{in} - \sum (\rho A V)_{out}$$



Let's now consider an infinitesimal control volume, stationary within the fluid flow. In 2D, this has size $dV = dx \cdot dy \cdot 1$.

The $\cdot$ 1 indicates multiplication by the control volume extension along z, taken as 1, and has units of length. It appears only for the units to be consistent; will be dropped in next slides.

Conservation of mass

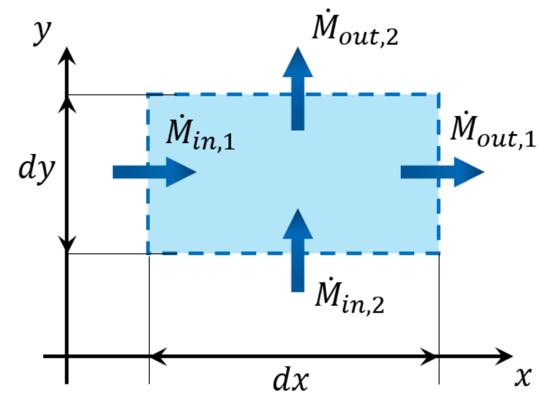
$$\frac{dM}{dt} = \dot{M}_{in,1} + \dot{M}_{in,2} - \dot{M}_{out,1} - \dot{M}_{out,2} \qquad units: \frac{kg}{s}$$

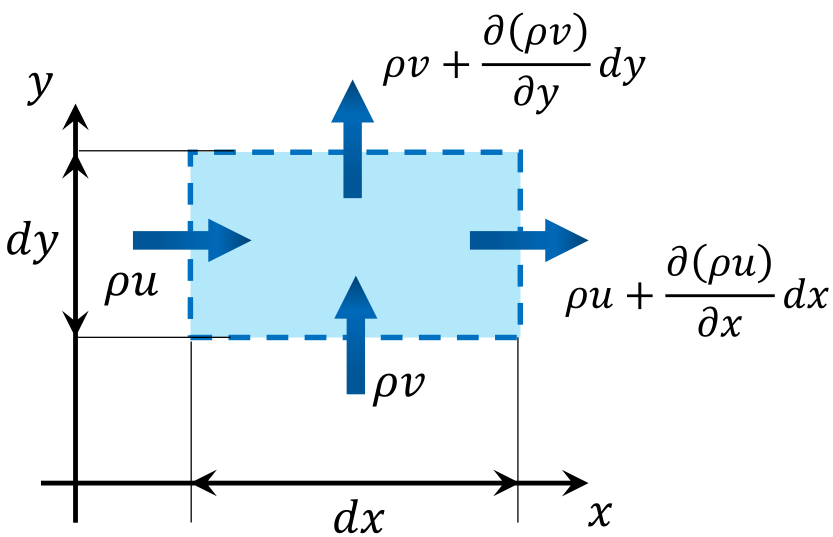
Let's now consider an infinitesimal control volume stationary within the fluid flow.

In 2D, this has size $dV = dx \cdot dy \cdot 1$.

Mass within the control volume: $M(x,y,t) = \rho(x,y,t)\, dx\, dy$

Temporal derivative becomes partial: $\frac{dM}{dt} \rightarrow \frac{\partial M}{\partial t} = \frac{\partial \rho}{\partial t} dx\, dy$





$$\dot{M}_{in,1} = \rho u\, dy \qquad \dot{M}_{in,2} = \rho v\, dx$$

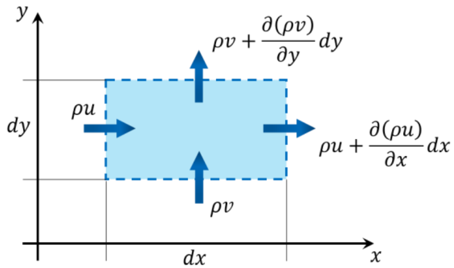
$$\dot{M}_{out,1} = \dot{M}_{in,1} + \frac{\partial \dot{M}_{in,1}}{\partial x} dx$$

$$= \left[\rho u + \frac{\partial(\rho u)}{\partial x} dx\right] dy$$

$$\dot{M}_{out,2} = \left[\rho v + \frac{\partial(\rho v)}{\partial y} dy\right] dx$$

Conservation of mass

$$\frac{dM}{dt} = \dot{M}_{in,1} + \dot{M}_{in,2} - \dot{M}_{out,1} - \dot{M}_{out,2}$$



Put everything together:

$$\frac{\partial \rho}{\partial t} dx\, dy = \cancel{\rho u\, dy} + \cancel{\rho v\, dx} - \left[\cancel{\rho u} + \frac{\partial(\rho u)}{\partial x} dx\right] dy - \left[\cancel{\rho v} + \frac{\partial(\rho v)}{\partial y} dy\right] dx$$

$$\Rightarrow \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y}\right] \cancel{dx\, dy} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad units: \frac{kg}{m^3 s}$$

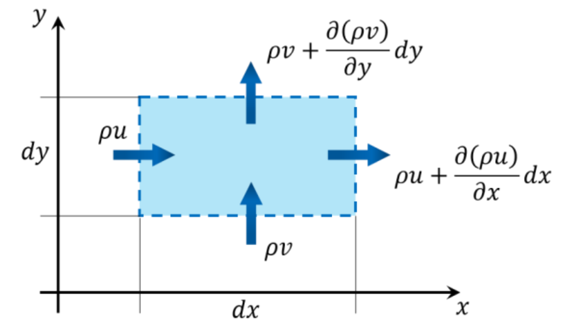
Equation of continuity: it holds at any point in space and time

Special case: **incompressible flow**

$$\rho = const \Rightarrow \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Equation of continuity for incompressible flow

Worked example 1



Consider the following steady (independent of time) 2D velocity field:

$$\boldsymbol{V}(x,y) = u(x,y)\hat{\boldsymbol{i}} + v(x,y)\hat{\boldsymbol{j}} = (2xy + 3y^2)\hat{\boldsymbol{i}} - y^2\hat{\boldsymbol{j}}$$

Is the flow incompressible?

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial v}{\partial y} = -2y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2y - 2y = 0$$

Yes, the flow is incompressible!

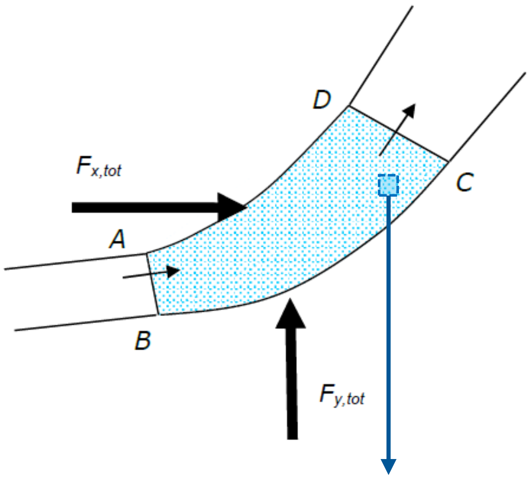
Conservation of momentum

Let's restart from TF1: lecture notes pages 99-100, linear momentum equation

$$\sum F_x = \sum \dot{Q}_{x,out} - \sum \dot{Q}_{x,in} + \frac{dQ_x}{dt}$$

$$\sum F_y = \sum \dot{Q}_{y,out} - \sum \dot{Q}_{y,in} + \frac{dQ_y}{dt}$$

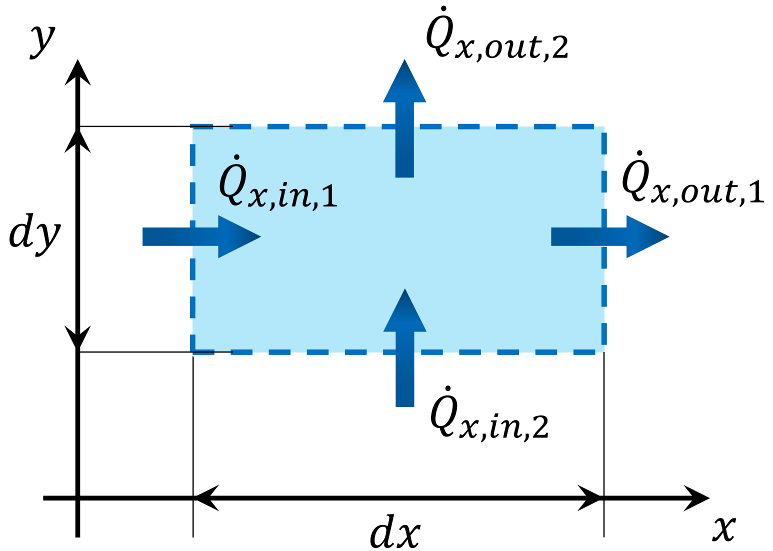
$units: N$



In TF2, we now consider the generic case of unsteady flow

Again, let's consider an infinitesimal control volume of size $dV = dx \cdot dy \cdot 1$, and focus on direction x. X-momentum in the control volume:

$$Q_x(x, y, t) = \rho u\, dx\, dy$$

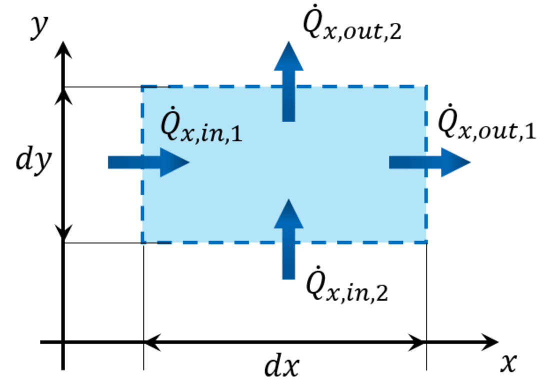


Temporal derivative becomes partial:

$$\frac{dQ_x}{dt} \rightarrow \frac{\partial Q_x}{\partial t} = \frac{\partial(\rho u)}{\partial t} dx\, dy$$

Conservation of momentum

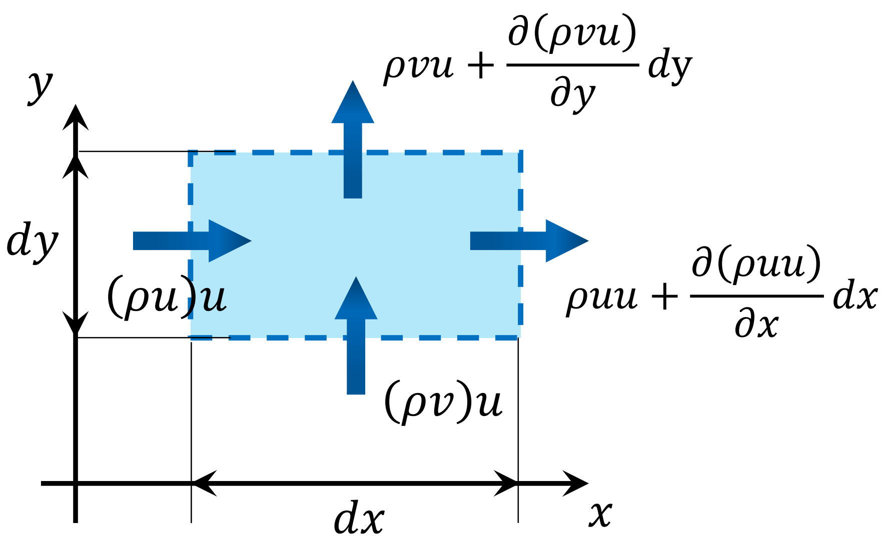
$$\sum F_x = \sum \dot{Q}_{x,out} - \sum \dot{Q}_{x,in} + \frac{dQ_x}{dt}$$



Again, let's consider an infinitesimal control volume of size $dV = dx \cdot dy \cdot 1$, and focus on direction x. X-momentum in the control volume: $Q_x(x,y,t) = \rho u \, dx \, dy$

Temporal derivative becomes partial:

$$\frac{dQ_x}{dt} \rightarrow \frac{\partial Q_x}{\partial t} = \frac{\partial(\rho u)}{\partial t} dx \, dy$$



$$\dot{Q}_{x,in,1} = \overbrace{(\rho u \, dy)}^{\dot{M}_{in,1}} u = \rho u u \, dy$$

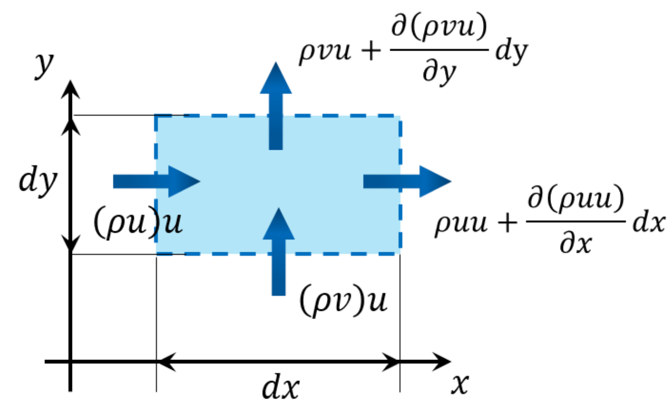
$$\dot{Q}_{x,in,2} = (\rho v \, dx) \, u = \rho v u \, dx$$

$$\dot{Q}_{x,out,1} = \left[\rho u u + \frac{\partial(\rho u u)}{\partial x} dx\right] dy$$

$$\dot{Q}_{x,out,2} = \left[\rho v u + \frac{\partial(\rho v u)}{\partial y} dy\right] dx$$

Conservation of momentum

$$\sum F_x = \sum \dot{Q}_{x,out} - \sum \dot{Q}_{x,in} + \frac{dQ_x}{dt}$$



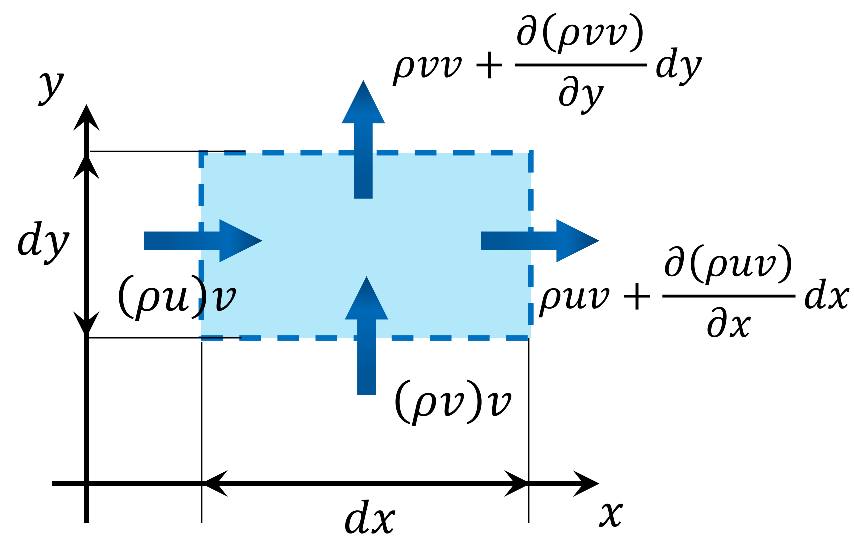
Put everything together:

$$\left[\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} + \frac{\partial(\rho v u)}{\partial y}\right] dx\, dy = \sum F_x$$

Repeating the same procedure for direction y:

$$\sum F_y = \sum \dot{Q}_{y,out} - \sum \dot{Q}_{y,in} + \frac{dQ_y}{dt}$$

$$\left[\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v v)}{\partial y}\right] dx\, dy = \sum F_y$$



Conservation of momentum

Momentum equation so far (still need to figure out the forces):

$$along\ x:\quad \left[\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u u)}{\partial x}+\frac{\partial(\rho v u)}{\partial y}\right]dx\,dy=\sum F_x$$

$$along\ y:\quad \left[\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho u v)}{\partial x}+\frac{\partial(\rho v v)}{\partial y}\right]dx\,dy=\sum F_y$$

Can also be rewritten by considering that, for example x-component:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho u u)}{\partial x}+\frac{\partial(\rho v u)}{\partial y}=\rho\frac{\partial u}{\partial t}+u\frac{\partial \rho}{\partial t}+\rho u\frac{\partial u}{\partial x}+u\frac{\partial(\rho u)}{\partial x}+\rho v\frac{\partial u}{\partial y}+u\frac{\partial(\rho v)}{\partial y}$$

$$=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+u\left(\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}\right)$$

=0, remember continuity!

$$along\ x:\quad \left[\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right]dx\,dy=\sum F_x$$

$$along\ y:\quad \left[\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\right]dx\,dy=\sum F_y$$

Forces in a fluid in motion

We need now to express the forces: which forces do act on a fluid in motion?

- Body forces, proportional to the mass in the control volume, for instance **gravity**:

$$F_{g,x} = Mg_x = \rho g_x dx\, dy$$

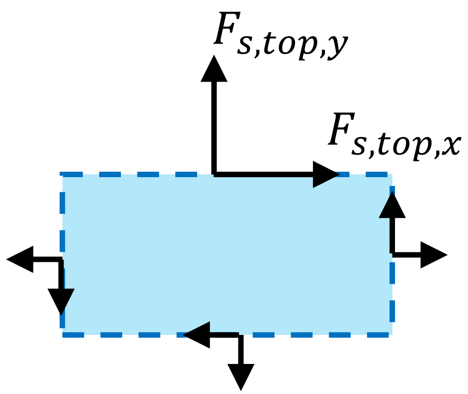
$$\boldsymbol{g} = g_x\hat{\boldsymbol{\imath}} + g_y\hat{\boldsymbol{\jmath}}$$

$$F_{g,y} = Mg_y = \rho g_y dx\, dy$$

- Surface forces $F_{s,x}$ and $F_{s,y}$, which are forces acting along the surfaces of the control volume and proportional to its surface area.

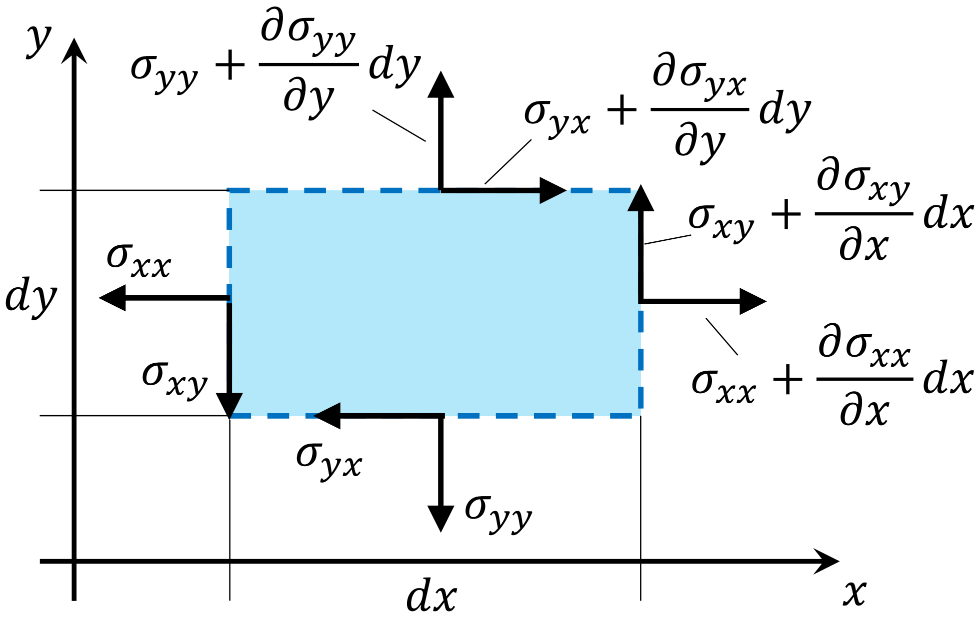
$$\sum F_x = F_{g,x} + F_{s,x} \qquad \sum F_y = F_{g,y} + F_{s,y}$$

How do we write down the overall surface force for our control volume? Each of the 4 surfaces is subjected to a normal and a tangential force, the figure shows for example the top surface



Representation of surface forces

We express the forces as *stresses*, that are forces per unit area (units: N/m^2=Pa)



σ_{ij}: surface stress acting on a face normal to the i axis, and directed along the j axis

$$F_{s,x} = -\sigma_{xx} dy - \sigma_{yx} dx + \left[\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right] dy + \left[\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy\right] dx = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right) dx\, dy$$

$$F_{s,y} = -\sigma_{xy} dy - \sigma_{yy} dx + \left[\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx\right] dy + \left[\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy\right] dx = \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right) dx\, dy$$

Note: it is not the stresses, but their <u>changes,</u> that cause a net force on the control volume

Representation of surface stresses

Therefore:

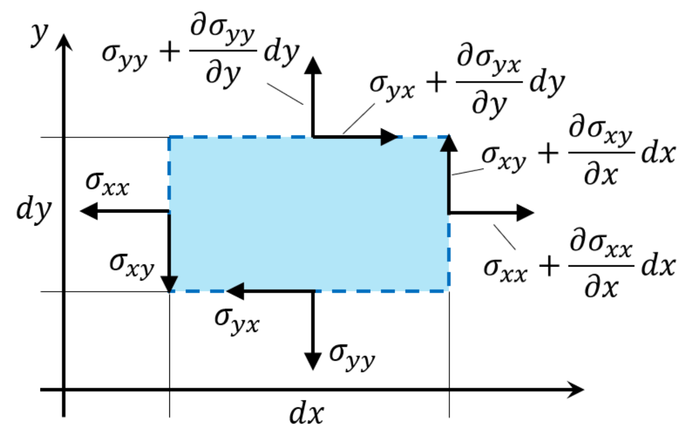
$$along\ x: \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dx\,dy = \sum F_x = F_{g,x} + F_{s,x} = \left[\rho g_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)\right]dx\,dy$$

$$along\ y: \quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)dx\,dy = \sum F_y = F_{g,y} + F_{s,y} = \left[\rho g_y + \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\right]dx\,dy$$

where the product $dx\,dy$ cancels out, leading to a first form of the momentum equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y + \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}$$



Note that:

σ_{xx}, σ_{yy}: stresses normal to the surface they apply to

σ_{xy}, σ_{yx}: stresses tangential to the surface they apply to

Representation of surface stresses

Our objective is to develop further the expressions for the surface stresses σ_{ij}.

The stresses acting on the surface of the control volume are the sum of:

- Hydrostatic pressure: always orthogonal to the surface and directed inward to the control volume, because it tends to compress it.
- Viscous (shear) stress: due to the motion of the fluid and the velocity gradients in it.

Therefore, we can rewrite:

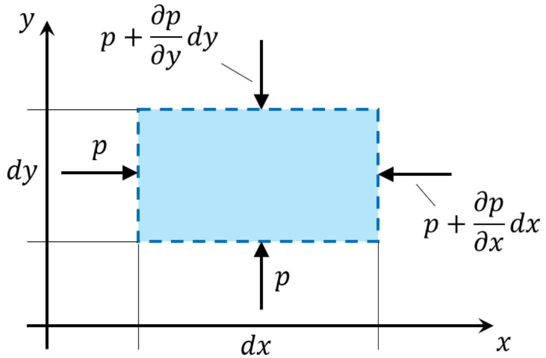
$$\sigma_{xx} = -p + \tau_{xx}$$

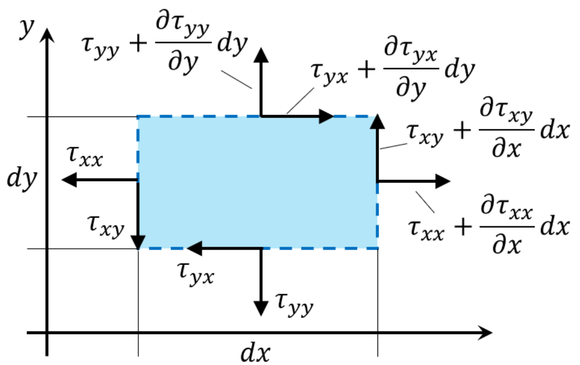
$$\sigma_{yy} = -p + \tau_{yy}$$

$\sigma_{xy} = \tau_{xy}$ (just a change of notation)

$\sigma_{yx} = \tau_{yx}$ (just a change of notation)

where pressure is negative because it is a compression force.





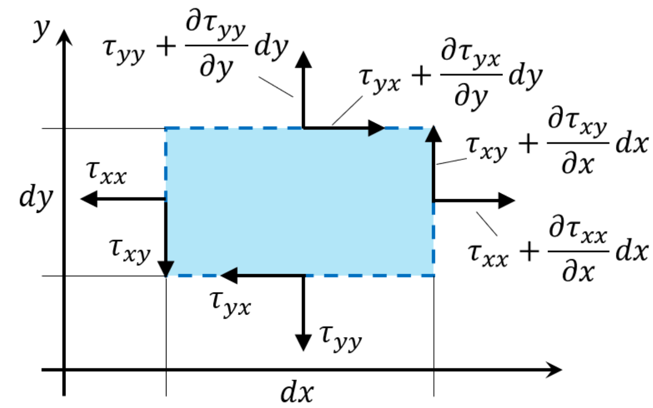
Representation of surface stresses

This takes us to a second form of the momentum equation:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}$$

Inertial force | Gravity force | Pressure force | Viscous force



However, this is not useful yet, because we do not know how to express the viscous stresses. In order to express them, we need a constitutive relation for the specific fluid, that tells us how the shear stress depends on the velocity field.

Most of the fluids encountered in the engineering practice (air, water, oil,…) exhibit a linear dependence between shear stress and shear rate (the velocity gradient). These fluids are called **Newtonian fluids**.

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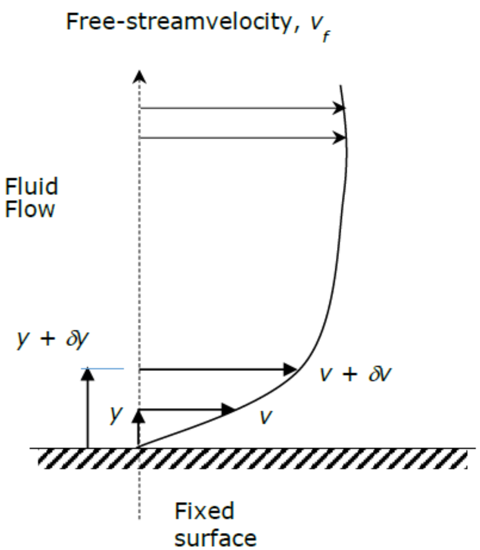
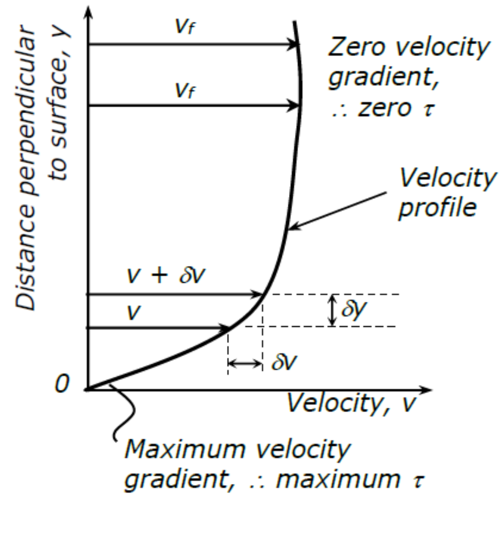
Newtonian fluids

From your TF1 notes (p. 64)





For most fluids used in engineering it is found that the shear stress τ is directly proportional to the velocity gradient when straight and parallel flow is involved. Thus:

$$\tau \propto \frac{dv}{dy} \qquad \text{or} \qquad \tau = \text{constant}\left(\frac{dv}{dy}\right)$$

The constant of proportionality is called the dynamic viscosity or often just the viscosity of the fluid and is denoted by μ. Hence:

$$\tau = \mu \frac{dv}{dy}$$

This is Newton's Law of Viscosity and fluids that obey it are known as Newtonian fluids. The equation is limited to straight and parallel (laminar) flow. Only if the flow is of this form does du represent the time rate of sliding of one layer over

Newtonian fluids

For Newtonian fluids, the dynamic viscosity $\mu\ [\mathrm{kg}/(m \cdot s)]$ is a property of the fluid and depends only on pressure and temperature.

In the case of a Newtonian fluid in incompressible flow, the constitutive relation takes the simple form:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

So that:

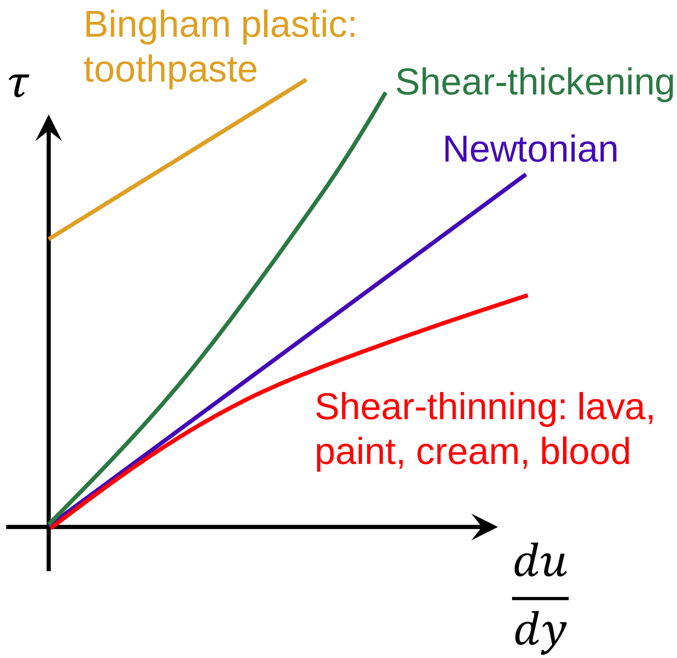
$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y}$$

$$\left.\begin{aligned}\tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\\ \tau_{yx} &= \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\end{aligned}\right\} \quad \tau_{xy} = \tau_{yx}$$

Non-Newtonian fluids

There are many interesting examples of non-Newtonian fluids:



Shear-thinning vs shear-thicknening

https://www.youtube.com/watch?v=X_cLJvUBlxw

Momentum equation for Newtonian fluid

This is where we were left:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

And we have seen that for a Newtonian fluid in incompressible flow:

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}, \qquad \tau_{yy} = 2\mu\frac{\partial v}{\partial y}, \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

We are now ready for our third and last version of the momentum equation. For the incompressible flow of a Newtonian fluid:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

The Navier-Stokes equations

We refer to the Navier–Stokes equations as the set of mass and momentum equations for a Newtonian fluid. In the case of incompressible flow, we have:

- Continuity equation: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

- Momentum equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

They form a system of 3 equations and 3 unknowns: p, u, v. If the flow is compressible, then the density is also unkwnown, and to close the system we need an equation of state $\rho = \rho(p, T)$, and an energy equation too.

The Navier-Stokes equations

More compact forms of the Navier-Stokes equations for incompressible flow.

We need to refresh some concepts from vector calculus:

Del operator (in 2D): $\nabla = \dfrac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \dfrac{\partial}{\partial y}\hat{\boldsymbol{\jmath}}$

Divergence of a vector, e.g. of $\boldsymbol{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}}$: $\nabla \cdot \boldsymbol{V} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$

Gradient of a scalar, e.g. of u and v: $\nabla u = \dfrac{\partial u}{\partial x}\hat{\boldsymbol{\imath}} + \dfrac{\partial u}{\partial y}\hat{\boldsymbol{\jmath}}$, $\quad \nabla v = \dfrac{\partial v}{\partial x}\hat{\boldsymbol{\imath}} + \dfrac{\partial v}{\partial y}\hat{\boldsymbol{\jmath}}$

Laplacian a scalar, e.g. of u and v: $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, $\quad \nabla^2 v = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \longrightarrow \quad \nabla \cdot \boldsymbol{V} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \longrightarrow \quad \rho\left(\frac{\partial u}{\partial t} + \boldsymbol{V}\cdot\nabla u\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\nabla^2 u$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad \longrightarrow \quad \rho\left(\frac{\partial v}{\partial t} + \boldsymbol{V}\cdot\nabla v\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\nabla^2 v$$

Boundary conditions

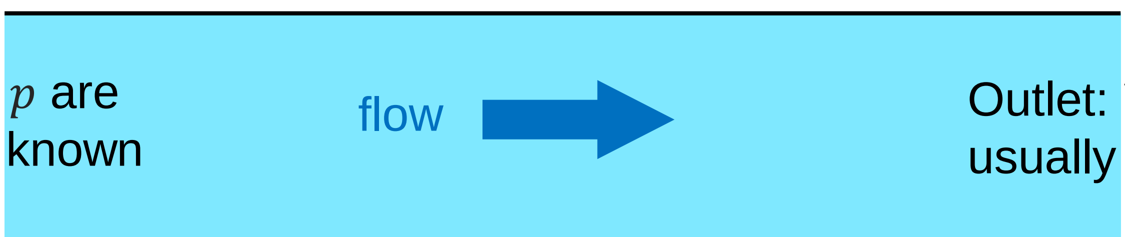
Which boundary conditions do apply to the N-S equations?

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Inlet: $\boldsymbol{V}, p$ are usually known



Outlet: $\boldsymbol{V}, p$ are usually known

Solid walls: no-slip condition, $\boldsymbol{V} = \boldsymbol{V}_{wall}$, which becomes $\boldsymbol{V} = 0$ if the wall is stationary

If the flow is unsteady, we need also initial conditions: $\boldsymbol{V}, p$ at $t = 0$.

Topic 1 – Navier-Stokes equations

What to take home from Topic 1

- How to apply the concept of control volume to derive conservation equations
- Continuity equation for compressible and incompressible flows
- How to express body and surface forces acting on a fluid in motion
- General form of the momentum equation
- Constitutive law for Newtonian fluids and related momentum equation
- Newtonian fluids and incompressible flow: the Navier-Stokes equations

Further reading:

- F. White book, Sec. 4.1, 4.2, 4.3, 4.6
- F. White book, Sec. 4.9: Euler's equation and Bernoulli theorem

Seminar

Analytical solution of the N-S equations

The analytical solution of the N-S equations is available only for a few simplified flow configurations.

Demonstration of existence and uniqueness of the solution of the Navier-Stokes equation in 3D is one of the seven **Millennium Problems**:

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

http://www.claymath.org/millennium-problems

The Clay Mathematics Institute (CMI) has named seven "Millennium Prize Problems." The Scientific Advisory Board of CMI (SAB) selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solutions to these problems, with $1 million allocated to each. The Directors of CMI, and no other persons or body, have the authority to authorize payment from this fund or to modify or interpret these stipulations. The Board of Directors of CMI makes all mathematical decisions for CMI, upon the recommendation of its SAB.

Official problem statement:

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t), u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence-free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

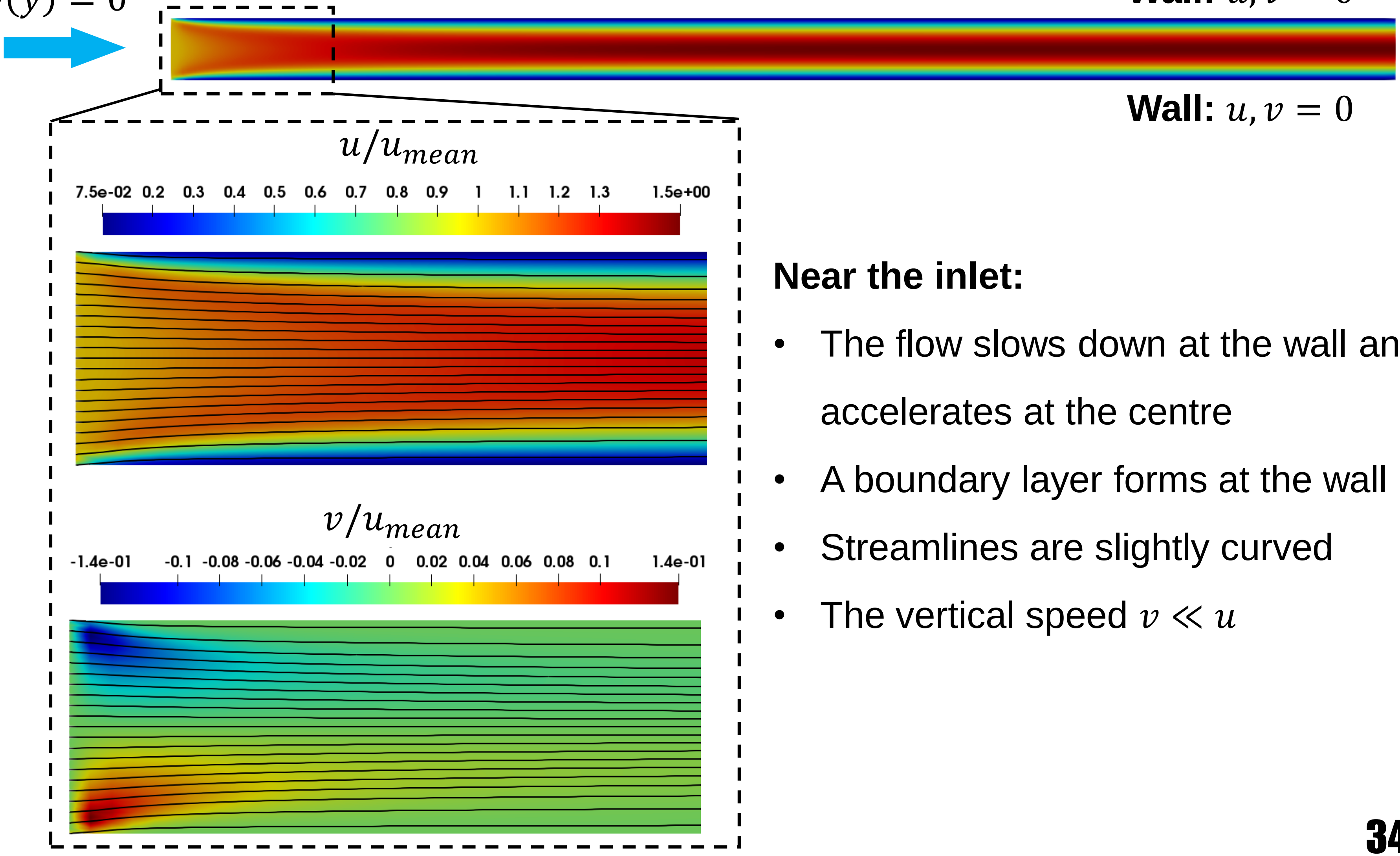
(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence-free vector field $u^\circ(x)$ on $\mathbb{R}^3$ and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

Steady flow within a 2D channel

Inlet:

$u(y) = u_{mean}$

$v(y) = 0$



Near the inlet:

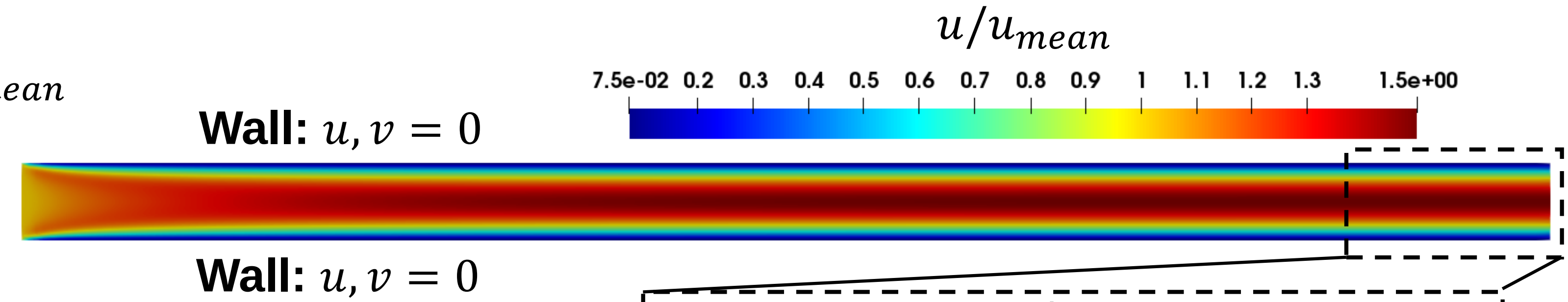
- The flow slows down at the wall and accelerates at the centre
- A boundary layer forms at the wall
- Streamlines are slightly curved
- The vertical speed $v \ll u$

Steady flow within a 2D channel

Inlet:

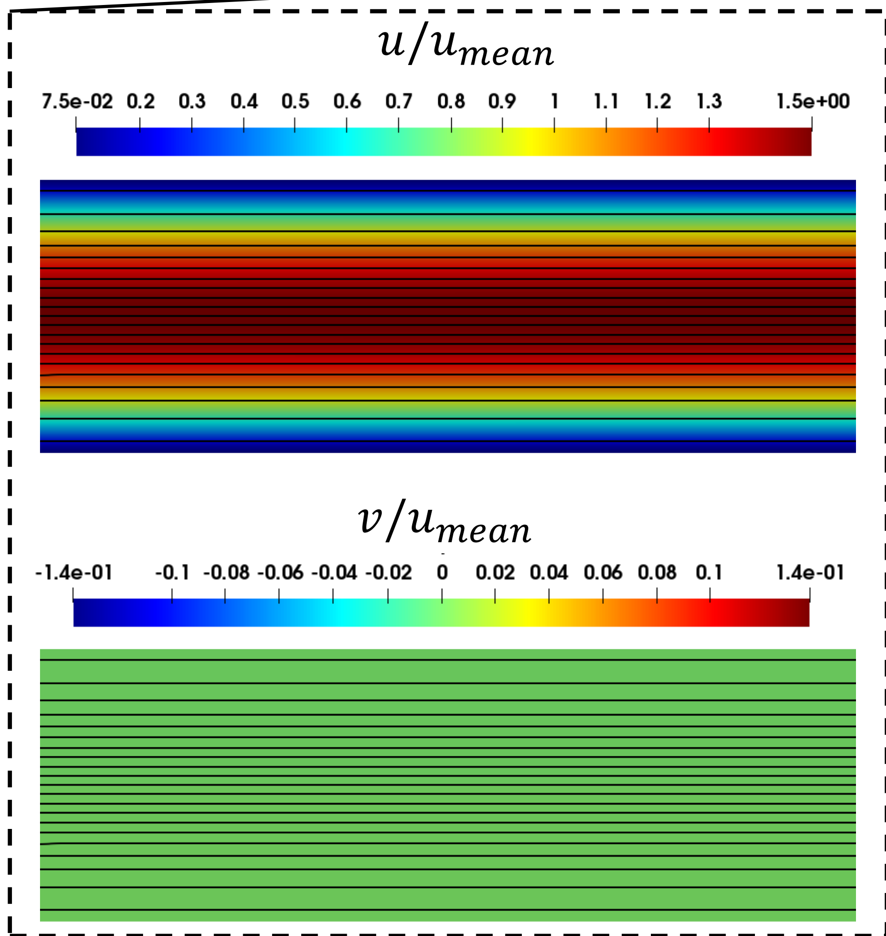
$u(y) = u_{mean}$

$v(y) = 0$



Far from the inlet:

- The flow does not change anymore along x. We say that the flow is <u>fully-developed</u>
- Streamlines are perfectly horizontal, it means that the flow is uni-directional along x
- The vertical speed v is exactly zero

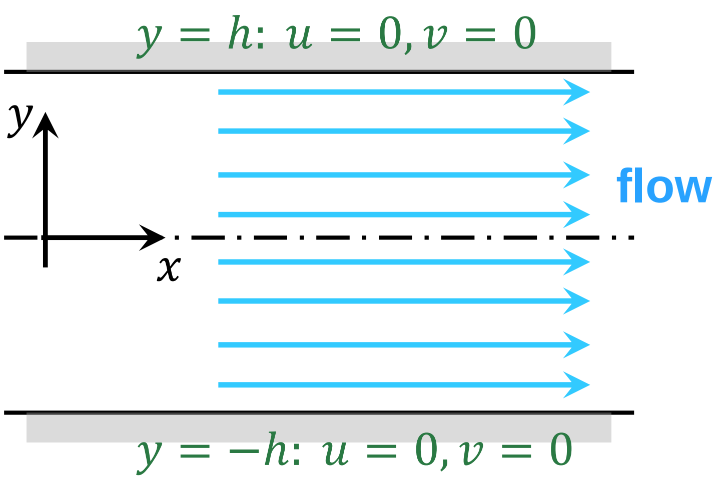


The unidirectional flow far from the inlet has analytical solution (will see)

Worked example 2

Consider the incompressible flow of a Newtonian fluid in the following conditions:

- The fluid flows between two stationary, infinitely extended parallel walls
- The flow is at steady-state, thus all time derivatives are zero
- Far from the inlet, the streamlines are horizontal (slide 34) and $v = 0$ everywhere
- The flow is driven by a constant streamwise pressure gradient $\partial p/\partial x$
- The gravity force is neglected, $\boldsymbol{g} = 0$



These simplify the N-S equations into:

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} = 0$$

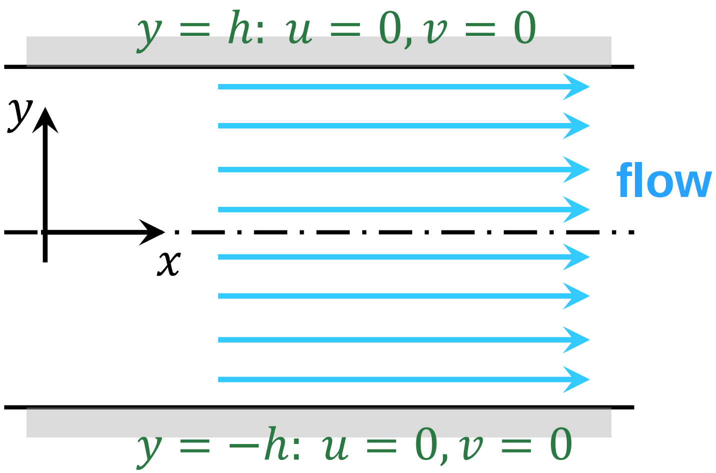
$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}\right) = \cancel{\rho g_x} - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + u\cancel{\frac{\partial v}{\partial x}} + v\cancel{\frac{\partial v}{\partial y}}\right) = \cancel{\rho g_y} - \frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

Worked example 2

What is left? From continuity:

$$\frac{\partial u}{\partial x} = 0 \Longrightarrow \boxed{u = u(y)\ only}$$



From the y-momentum equation:

$$\frac{\partial p}{\partial y} = 0 \Longrightarrow \boxed{p = p(x)\ only}$$

X-momentum equation:

$$\rho u \frac{\partial u}{\partial x} = -\frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{d^2 u}{d y^2}\right) \longrightarrow \boxed{\mu \frac{d^2 u}{d y^2} = \frac{dp}{dx}}$$

To be solved with boundary conditions: $u = 0$ for $y = +h, -h$

Note. The relation $\tau = \mu\, du/dy$ holds as long as the flow is laminar (will see this in T2):

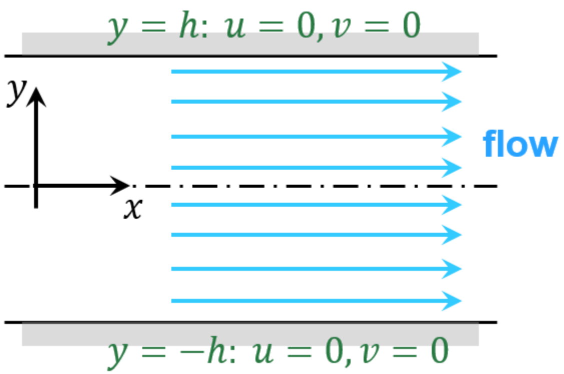
$$Re = \frac{\rho u_{mean} \ell}{\mu} < 2000$$

$\ell = 4h$: characteristic length for flow between parallel plates

Therefore, the solution derived next is correct only for a laminar flow

Worked example 2

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$



We integrate twice with respect to y:

$$u(y) = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

And obtain the constants by imposing the no-slip condition at the walls:

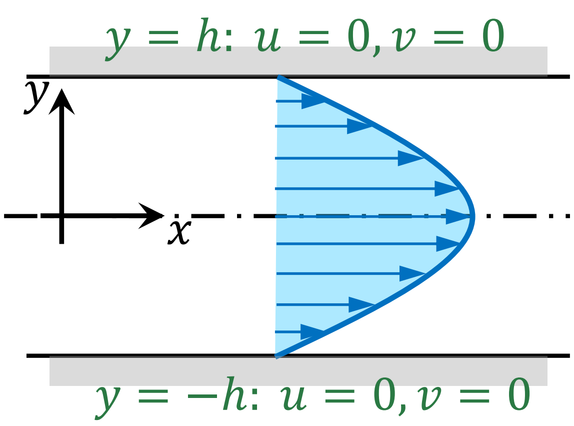
$$u(y = +h) = 0 \Longrightarrow \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} + C_1 h + C_2 = 0$$

$$u(y = -h) = 0 \Longrightarrow \frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2} - C_1 h + C_2 = 0$$

$$C_1 = 0, \qquad C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

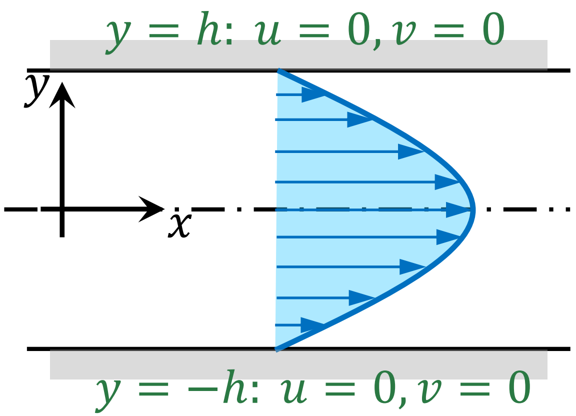
$$u(y) = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$$

Parabolic velocity profile



Note: $u > 0$ because $dp/dx < 0$, as the pressure decreases along the channel due to the wall shear.

Worked example 2



Max speed at the channel centre (y=0):

$$u_{max} = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Volumetric flow rate: $\dot{V} = \int_{-h}^{h} u(y)dy = -\frac{dp}{dx}\frac{2h^3}{3\mu}$

And, for a channel of length L: $\Delta p = \frac{3\mu\dot{V}L}{2h^3}$ total pressure drop in the channel

Average flow velocity: $u_{mean} = \frac{1}{2h}\int_{-h}^{h} u(y)dy = -\frac{dp}{dx}\frac{h^2}{3\mu}$ → $u_{max} = 1.5\, u_{mean}$

Take a look back at the contours of u/u_{mean} in slide 34

We can rewrite: $u(y) = \frac{3}{2}u_{mean}\left(1 - \frac{y^2}{h^2}\right)$

Note: owing to the 2D geometry, we are neglecting the extension along z and the volumetric flow rate $\dot{V}$ is expressed in units of m^2/s.

Worked example 3

Repeat the derivation for a flow in a circular pipe of radius R and diameter $D = 2R$.

This time we need to use a cylindrical reference frame (r, ϑ, x). The Navier-Stokes equations, simplified as done in the previous example, tell us that:

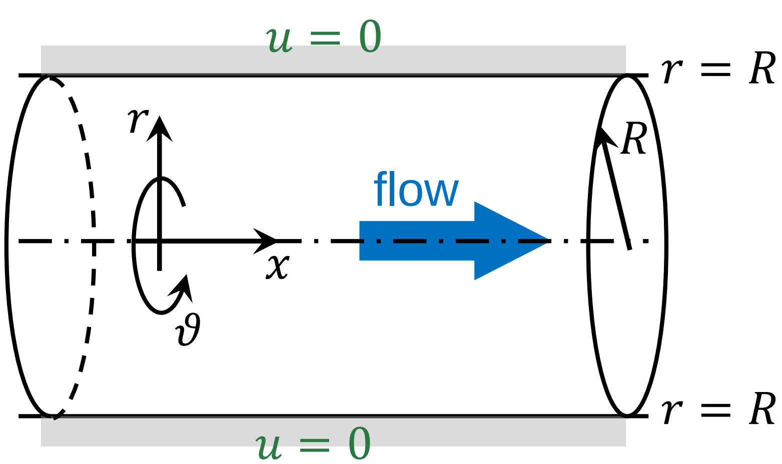
$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{dp}{dx}$$

To be now solved with boundary conditions:

$$u = 0, \qquad at\, r = R$$

$$\frac{du}{dr} = 0, \qquad at\, r = 0$$

The second condition requires that the flow has axial symmetry.



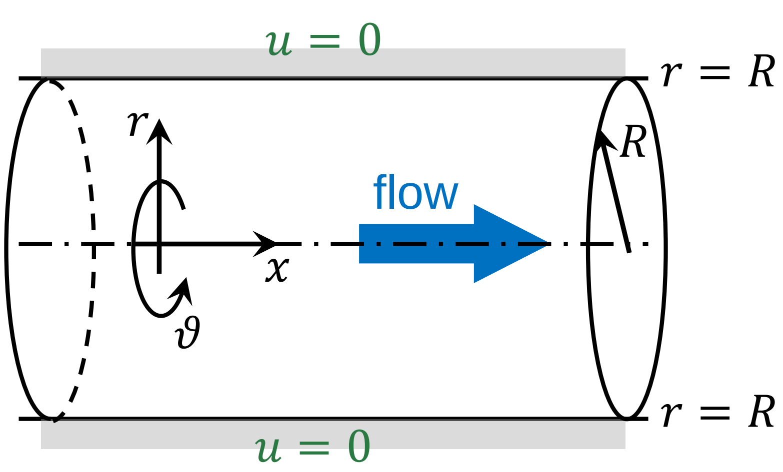
Worked example 3

We integrate twice with respect to r:

$$u(r) = \frac{dp}{dx}\frac{r^2}{4\mu} + C_1 \ln r + C_2$$

Axisymmetry condition: $\left.\frac{du}{dr}\right|_{r=0} = 0 \Longrightarrow \left(\frac{dp}{dx}\frac{r}{2\mu} + C_1\frac{1}{r}\right)_{r=0} = 0 \Longrightarrow C_1 = 0$

No-slip condition at the wall: $u(r = R) = 0 \Longrightarrow \frac{dp}{dx}\frac{R^2}{4\mu} + C_2 = 0 \Longrightarrow C_2 = -\frac{dp}{dx}\frac{R^2}{4\mu}$



$$u(r) = -\frac{dp}{dx}\frac{R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right)$$

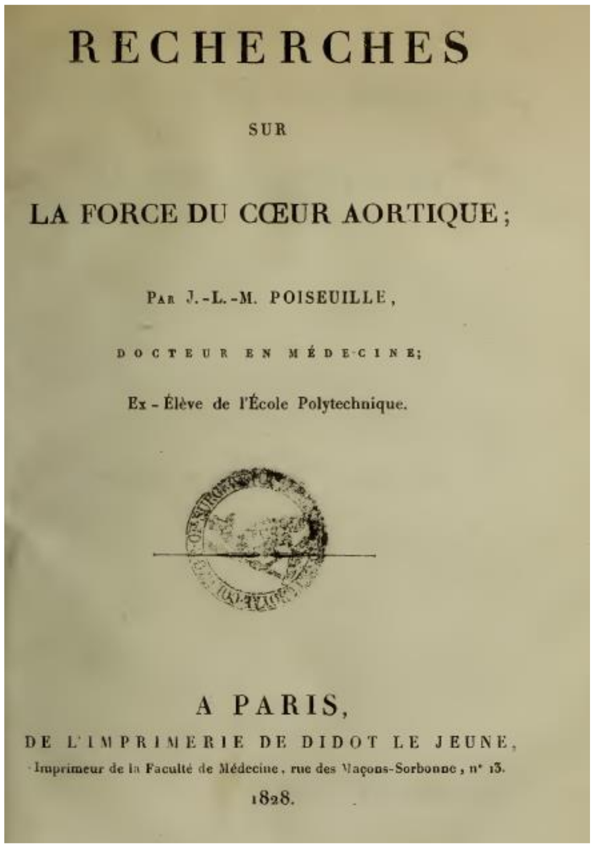
Parabolic velocity profile

Worked example 3

$$u(r) = -\frac{dp}{dx}\frac{R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right)$$

Also known as ‘Poiseuille’ velocity profile

Poiseuille was the first to derive (1828) the relationship between pipe length and diameter, flow rate, and pressure drop, as a model for blood flow in vessels.



RECHERCHES

SUR

LA FORCE DU CŒUR AORTIQUE;

PAR J.-L.-M. POISEUILLE,

DOCTEUR EN MÉDECINE;

Ex-Élève de l'École Polytechnique.

A PARIS,

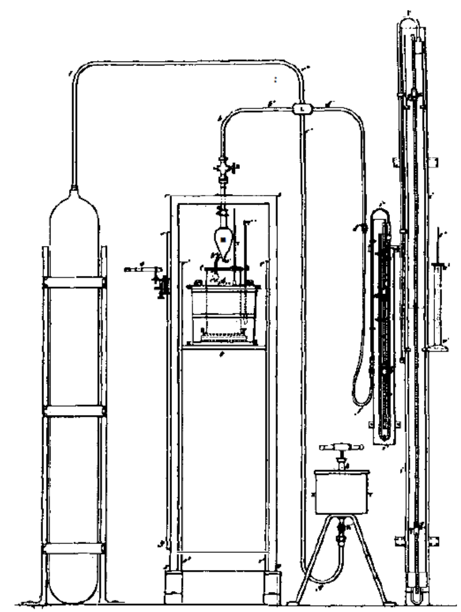
DE L'IMPRIMERIE DE DIDOT LE JEUNE,

Imprimeur de la Faculté de Médecine, rue des Maçons-Sorbonne, n° 13.

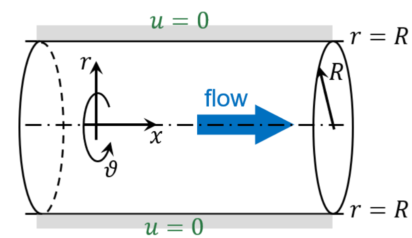
1828.

Poiseuille’s PhD thesis, https://ia800208.us.archive.org/22/items/b22291611/b22291611.pdf

Poiseuille’s experimental apparatus, https://www.annualreviews.org/doi/pdf/10.1146/annurev.fl.25.010193.000245



Worked example 3



Max speed at the channel centre (r=0):

$$u_{max} = -\frac{dp}{dx}\frac{R^2}{4\mu}$$

Volumetric flow rate: $\dot{V} = \int_0^R u(r)2\pi r\,dr = -\frac{dp}{dx}\frac{\pi R^4}{8\mu}$

Note: the volumetric flow rate $\dot{V}$ is now in m^3/s.

And, for a channel of length L: $\Delta p = \frac{8\mu\dot{V}L}{\pi R^4}$ total pressure drop in the channel

Average flow velocity: $u_{mean} = \frac{\dot{V}}{\pi R^2} = -\frac{dp}{dx}\frac{R^2}{8\mu}$ → $u_{max} = 2u_{mean}$

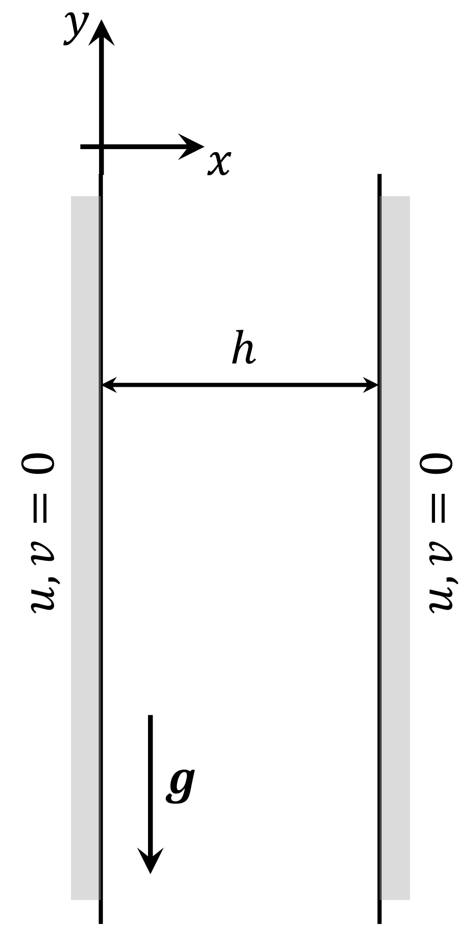
Wall shear stress: $\tau_w = \mu \left.\frac{du}{dr}\right|_{r=R} = -\frac{dp}{dx}\frac{R}{2}$

Friction factor: $f = \frac{8\tau_w}{\rho u_{mean}^2} = \frac{64}{Re}, \quad with \quad Re = \frac{\rho u_{mean} D}{\mu}$

Skin friction coefficient: $C_f = \frac{2\tau_w}{\rho u_{mean}^2} = \frac{f}{4} = \frac{16}{Re}$

Worked example 4

Air is flowing vertically between two stationary, infinitely extended parallel plates. The distance between the walls is $h = 0.02\ m$. Gravity acts vertically downward with magnitude $g_y = -9.81\ m/s^2$. The flow is subjected to a constant streamwise pressure gradient $\partial p/\partial y$. Assuming that the flow is steady, laminar and fully-developed, and that the geometry is 2D, (a) determine the velocity profile of air between the plates, (b) calculate the necessary streamwise pressure gradient $\partial p/\partial y$ in order to achieve a vertical upward flow rate of air of $\dot{V} = 0.008\ m^2/s$, and (c) the resulting shear stress at the wall. For air, use ρ=1 kg/m³ and μ=0.000018 kg/(m·s). Explain all the assumptions you make.



Note: owing to the 2D geometry, we neglect the extension along z and the flow rate is expressed in units of m^2/s.

Worked example 4

Solution

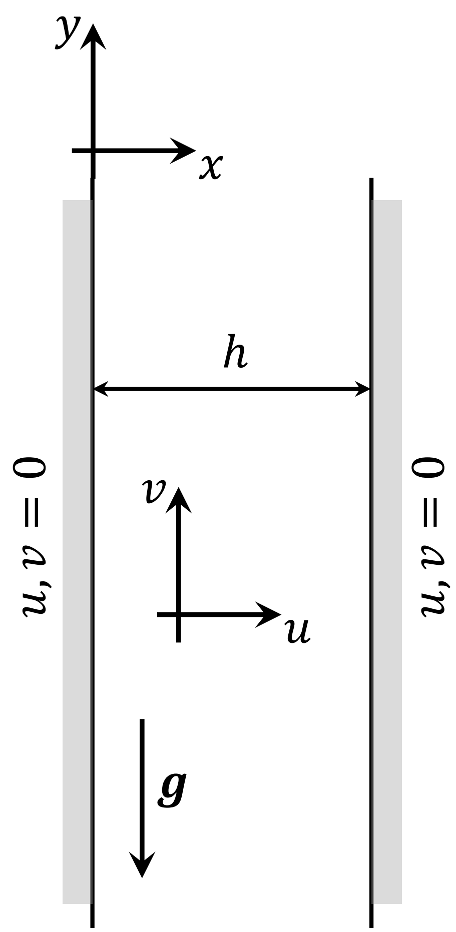
(a) Assumptions:

- Steady flow
- Fully-developed flow: $u = 0, v = v(x)$ only.
- Gravity is only vertical, $g_x = 0$.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$



Worked example 4

The x-momentum equation tells us that:

$$\frac{\partial p}{\partial x} = 0 \Longrightarrow p = p(y)\ only$$

From the y-momentum equation: $\rho g_y - \frac{dp}{dy} + \mu \frac{d^2 v}{dx^2} = 0$

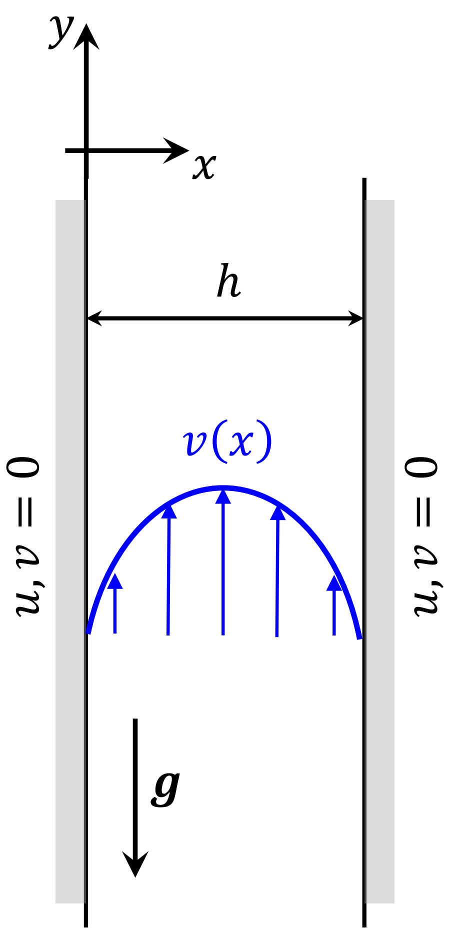
Integrate twice along x: $v(x) = \frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)x^2 + C_1 x + C_2$

Boundary conditions:

$$v(x = 0) = 0 \Longrightarrow C_2 = 0$$

$$v(x = h) = 0 \Longrightarrow \frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)h^2 + C_1 h = 0$$

$$\Longrightarrow C_1 = -\frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)h$$



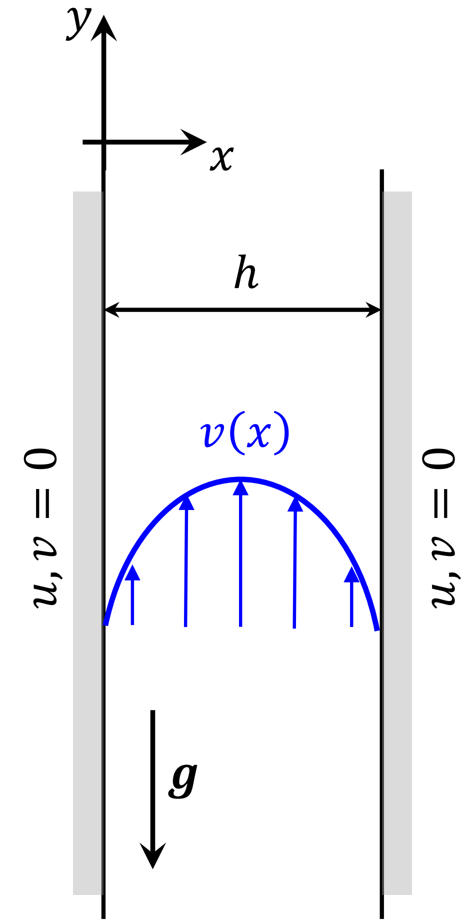
$$v(x) = \frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)(x^2 - hx)$$

Note: $(x^2 - hx) < 0$, and therefore we need $\left(\frac{dp}{dy} - \rho g_y\right) < 0$ for upward flow ($v > 0$)

Worked example 4

(b) We first need to express the volumetric flow rate based on the newly found velocity profile:

$$\dot{V} = \int_0^h v(x)dx = -\frac{h^3}{12\mu}\left(\frac{dp}{dy} - \rho g_y\right)$$



Rearrange to express the pressure gradient:

$$\frac{dp}{dy} = \rho g_y - \frac{12\mu\dot{V}}{h^3}$$

Therefore:

$$\frac{dp}{dy} = 1\,\frac{kg}{m^3} * \left(-9.81\,\frac{m}{s^2}\right) - \frac{12 * 18 * 10^{-6}\,\frac{kg}{m*s} * 0.008\,\frac{m^2}{s}}{(0.02\ m)^3} = -10\,\frac{Pa}{m}$$

Worked example 4

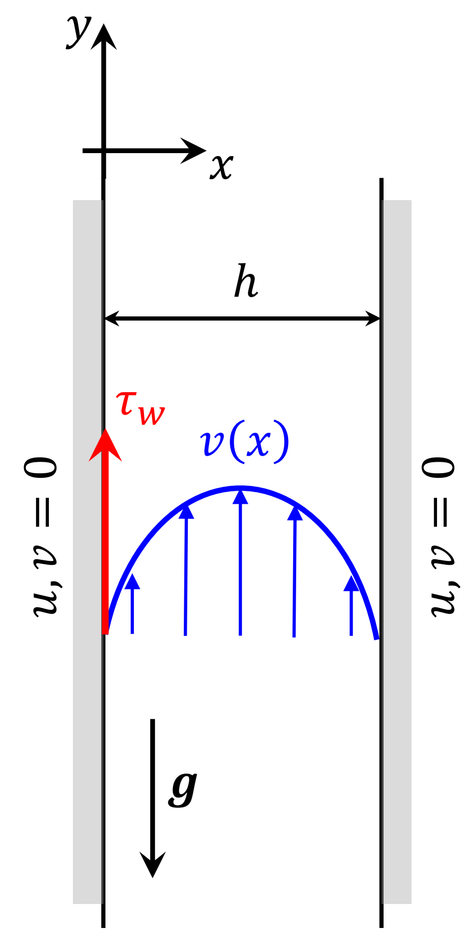
(c) Shear stress exerted by the fluid on the wall:

$$\tau_w = \mu \left.\frac{dv}{dx}\right|_{x=0} = \mu \left[\frac{1}{2\mu}\left(\frac{dp}{dy} - \rho g_y\right)(2x - h)\right]_{x=0}$$

$$= -\frac{h}{2}\left(\frac{dp}{dy} - \rho g_y\right)$$

$$= -\frac{0.02\ m}{2}\left(-10\,\frac{Pa}{m} - 1\,\frac{kg}{m^3} * \left(-9.81\,\frac{m}{s^2}\right)\right)$$

$$= 0.002\ Pa$$



Worked example 5

A SAE 10W oil flows at 1.1 m^3/h through a horizontal pipe with d=2 cm and L=12 m.

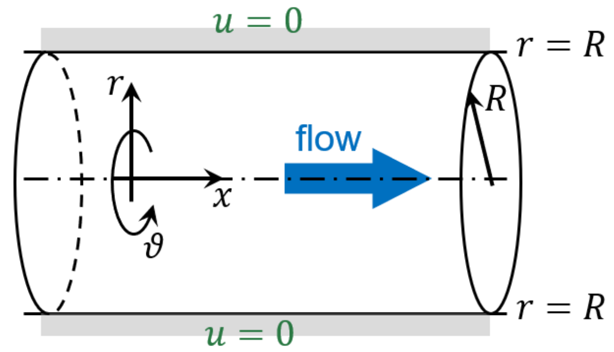
Find (a) the average velocity, (b) the Reynolds number, (c) the pressure drop and (d) the pumping power required. For the oil, use ρ=870 kg/m^3 and μ=0.104 kg/(m·s).

Use the theory for laminar flow within a circular pipe seen in WE3.

World's first beer pipeline:
https://logisticsmgepsupv.wordpress.com/2019/04/10/facilitating-transportation-systems-the-worlds-first-beer-pipeline/

Worked example 5

Solution



(a)

$$\dot{V} = 1.1\,\frac{m^3}{h} = 3.06 * 10^{-4}\,\frac{m^3}{s}$$

$$u_{mean} = \frac{\dot{V}}{\pi R^2} = \frac{3.06 * 10^{-4}\,\frac{m}{s}}{\pi(0.01\,m)^2} = 0.974\,\frac{m}{s}$$

(b)

$$Re = \frac{\rho u_{mean} D}{\mu} = \frac{870\,\frac{kg}{m^3} * 0.974\,\frac{m}{s} * 0.02\,m}{0.104\,\frac{kg}{m * s}} = 163$$

(c)

$$\Delta p = \frac{8\mu\dot{V}L}{\pi R^4} = \frac{8 * 0.104\,\frac{kg}{m * s} * 3.06 * 10^{-4}\,\frac{m^3}{s} * 12\,m}{\pi(0.01\,m)^4} = 97247\,Pa$$

(d)

$$Power = \dot{V}\Delta p = 3.06 * 10^{-4}\,\frac{m^3}{s} * 97247\,Pa = 29.8\,W$$

Worked example 6

From 2020/21 January exam

8. Consider the incompressible two-dimensional flow of a Newtonian fluid in the following conditions:

- the fluid flows between two horizontal, parallel, infinitely extended walls, with the bottom wall being stationary and the top wall translating with speed U_w in the positive x direction, see the Figure Q8 below;
- the flow is steady-state;
- we are considering a section of the duct far from the inlet, and thus the streamlines of the flow are horizontal;
- the flow is subjected to a constant streamwise pressure gradient $\partial p/\partial x$;
- the gravitational force on the flow is negligible.

Starting with the 2D incompressible Navier-Stokes equations, use the information above to produce the theoretical velocity profile in the duct, $u(y)$. Given the pressure gradient is 5 $Pa{\cdot}m^{-1}$, the wall velocity is 0.5 $m{\cdot}s^{-1}$, the plate separation is 10 mm and the viscosity of the fluid is 0.001 $kg{\cdot}m^{-1}{\cdot}s^{-1}$, calculate the velocity at y = 7 mm [4]

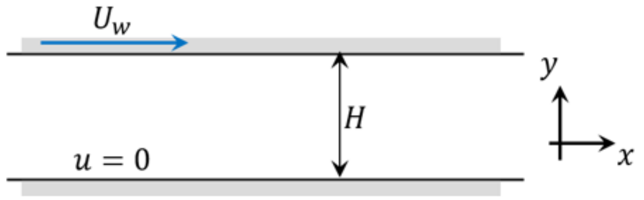


Figure Q8: Sketch of the flow configuration for Q8.

Worked example 6

Solution

Under the assumptions listed, the x-momentum equation reduces to:

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrated twice: $u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + ay + b$

Constants found by imposing $u(y=0) = 0, u(y=H) = U_w$: $a = \frac{U_w}{H} - \frac{1}{\mu}\frac{dp}{dx}\frac{H}{2}, b = 0$

The resulting theoretical profile is: $u(y) = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - Hy) + \frac{U_w}{H}y$

Flow velocity at y=7 mm:

$$u = \frac{1}{2 \times 0.001\ \frac{kg}{m \times s}}\left(-5\ \frac{Pa}{m}\right)\left((0.007\ m)^2 - 0.01\ m \times 0.007\ m\right) + \frac{0.5\ \frac{m}{s}}{0.01\ m} \times 0.007\ m$$

$$= 0.053\, m/s + 0.350\ m/s = 0.403\, m/s$$

Analytical solution of the N-S equations

Further reading/assessment:

- F. White book, Sec. 4.10
- F. White book, exercises 4.79, 4.80, 4.81,4.82, 4.83, 4.84, 4.86,4.87, 4.88, 4.89, 4.90, 4.92, 4.93, 4.94, 4.95.